JUMPS OF TERNARY CYCLOTOMIC COEFFICIENTS

BARTŁOMIEJ BZDĘGA

ABSTRACT. It is known that two consecutive coefficients of a ternary cyclotomic polynomial $\Phi_{pqr}(x) = \sum_k a_{pqr}(k)x^k$ differ by at most one. In this paper we give a criterion on k to satisfy $|a_{pqr}(k) - a_{pqr}(k-1)| = 1$. We use this to prove that the number of nonzero coefficients of the nth ternary cyclotomic polynomial is greater than $n^{1/3}$.

1. INTRODUCTION

We define the nth cyclotomic polynomial in the following way

$$\Phi_n(x) = \prod_{1 \leq m \leq n,\ (m,n)=1} (x - \zeta_n^m) = \sum_{k \in \mathbb{Z}} a_n(k)x^k, \quad \text{where } \zeta_n = e^{2\pi i/n}.$$

We say that a polynomial Φ_n is binary if n is a product of two distinct odd primes, ternary if n is a product of three distinct odd primes, etc.

The coefficients of cyclotomic polynomials are an interesting object to study. One of the most intensively studied research directions is estimating the maximal absolute value of a coefficient of Φ_n [1, 3, 4, 5, 6, 8]. There are also papers on the sum of the absolute values of coefficients of Φ_n [3, 8] and on the number θ_n of its nonzero coefficients [7, 9].

Ternary cyclotomic polynomials have an interesting property discovered by Gallot and Moree [10]: the difference between $a_{pqr}(k)$ and $a_{pqr}(k-1)$ never exceeds 1. In this paper, for a given ternary cyclotomic polynomial Φ_{pqr}, we characterize all k such that $|a_{pqr}(k) - a_{pqr}(k-1)| = 1$. Also we determine the number of k's for which this equality holds.

We say that the coefficient $a_{pqr}(k)$ is jumping up if $a_{pqr}(k) = a_{pqr}(k-1) + 1$. Analogously we define jumping down coefficients. Cyclotomic polynomials are known to be palindromic, i.e. $a_n(k) = a_n(\varphi(n) - k)$, where $\varphi(n)$ is the Euler function and the degree of Φ_n. Therefore the number of jumping up and the number of jumping down coefficients are equal and we denote this number by J_{pqr}.

One of our main results is the following theorem.

Theorem 1.1. *For a ternary cyclotomic polynomial Φ_n we have $J_n > n^{1/3}$.*

One half of the total number of jumping (up or down) coefficients is the lower bound for the number of odd coefficients of Φ_{pqr} and thus it is the lower bound for the number θ_{pqr} of nonzero coefficients of Φ_{pqr}. So we have

Corollary 1.2. *Let Φ_n be a ternary cyclotomic polynomial. Then $\theta_n > n^{1/3}$.*

We do not know if for every $\epsilon > 0$ there exist infinite classes of ternary cyclotomic polynomials Φ_n with $J_n < n^{1/3+\varepsilon}$. However, under some strong assumptions, we can prove that they do.

Theorem 1.3. *Let $\varepsilon > 0$. If q is a Germain prime, $q + 1$ has a prime divisor $p > q^{1-\varepsilon}$ and $r = 2q + 1$, then $J_n < 10n^{1/(3-\varepsilon)}$, where $n = pqr$.*

If the celebrated Schinzel Hypothesis H is true then there exist infinitely many triples of primes (p, q, r) satisfying conditions of Theorem 1.3. For example, we can put $(p, q, r) = (m, 6m - 1, 12m - 1)$.

The paper is organized in the following way. In the second section we recall some results form our earlier work [6]. In the third section we give a criterion on k determining $V(k) = a_{pqr}(k) - a_{pqr}(k - 1) \in \{-1, 0, 1\}$. In the fourth section we derive a formula on J_{pqr} and prove Theorem 1.1. In the fifth section we prove Theorem 1.3 and discuss the case of inclusion-exclusion polynomials.

2. Preliminaries

Troughout the paper we fix distinct odd primes p, q, r. Let us emphasize that every fact we prove for (p, q, r) has also an appropriate symmetric versions.

By $a^{-1}(b)$ we denote the inverse of a modulo b for $(a, b) = 1$. We treat this number as an integer from the set $\{1, 2, \ldots, b - 1\}$.

For every integer k we define $F_k \in \mathbb{Z}$ and $a_k \in \{0, 1, \ldots, p - 1\}$, $b_k \in \{0, 1, \ldots, q - 1\}$, $c_k \in \{0, 1, \ldots, r - 1\}$ by the equation

$$k + F_k pqr = a_k qr + b_k rp + c_k pq,$$

which clearly has the unique solution (F_k, a_k, b_k, c_k) depending on k. In [6] we proved the following properties of numbers F_k.

Proposition 2.1 ([6], a remark before Lemma 2.1). *For $-(qr + rp + pq) < k < pqr$ we have $F_k \in \{0, 1, 2\}$.*

Proposition 2.2 ([6], Lemma 2.2). *We have*

$$F_k - F_{k-q} = \begin{cases} -1, & \text{if } a_k < r^{-1}(p) \text{ and } c_k < p^{-1}(r), \\ 1, & \text{if } a_k \geq r^{-1}(p) \text{ and } c_k \geq p^{-1}(r), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.3 ([6], Lemma 2.3). *We have*

$$F_k - F_{k-q} - F_{k-r} + F_{k-q-r} = \begin{cases} -1, & \text{if } a_k \in \mathcal{A}_1^p, \\ 1, & \text{if } a_k \in \mathcal{A}_3^p, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\mathcal{A}_1^p = \{0, 1, \ldots, p-1\} \cap \left[q^{-1}(p) + r^{-1}(p) - p, \min\{q^{-1}(p), r^{-1}(p)\} \right),$$
$$\mathcal{A}_3^p = \{0, 1, \ldots, p-1\} \cap \left[\max\{q^{-1}(p), r^{-1}(p)\}, q^{-1}(p) + r^{-1}(p) \right).$$

Proposition 2.4 ([6], Lemma 2.4). *We have*

$$F_k - F_{k-p} - F_{k-q} - F_{k-r} + F_{k-q-r} + F_{k-r-p} + F_{k-p-q} - F_{k-p-q-r} = 0.$$

Proposition 2.5 ([6], Lemma 5.1). *For $k = 0, 1, \ldots, pqr - 1$ we have*

$$
\begin{aligned}
a_{pqr}(k) &- a_{pqr}(k-1) \\
&= N_0(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}) - N_0(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) \\
&= N_2(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}) - N_2(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) \\
&= \frac{1}{2}\left(N_1(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) - N_1(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}) \right),
\end{aligned}
$$

where $N_t(s)$ denotes the number of t's in the sequence (s).

3. A CRITERION ON JUMPING COEFFICIENTS

We define five following sets:

$$
\begin{aligned}
\mathcal{A}_0^p &= \{0, 1, \ldots, p-1\} \cap \left[0, q^{-1}(p) + r^{-1}(p) - p \right), \\
\mathcal{A}_1^p &= \{0, 1, \ldots, p-1\} \cap \left[q^{-1}(p) + r^{-1}(p) - p, \min\{q^{-1}(p), r^{-1}(p)\} \right), \\
\mathcal{A}_2^p &= \{0, 1, \ldots, p-1\} \cap \left[\min\{q^{-1}(p), r^{-1}(p)\}, \max\{q^{-1}(p), r^{-1}(p)\} \right), \\
\mathcal{A}_3^p &= \{0, 1, \ldots, p-1\} \cap \left[\max\{q^{-1}(p), r^{-1}(p)\}, q^{-1}(p) + r^{-1}(p) \right), \\
\mathcal{A}_4^p &= \{0, 1, \ldots, p-1\} \cap \left[q^{-1}(p) + r^{-1}(p), p \right).
\end{aligned}
$$

Similarly we define $\mathcal{A}_j^q$ and $\mathcal{A}_j^r$ for $j = 0, 1, 2, 3, 4$.

By Preposition 2.5, we have to consider 8-tuples

$$\text{oct}(k) = (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}, F_{k-p-q-r}).$$

We write $\text{oct}(k) \sim (t_1, \ldots, t_8)$ if $\text{oct}(k) = (t_1 + u, \ldots, t_8 + u)$ for some integer u. Put also

$$V(k) = a_{pqr}(k) - a_{pqr}(k-1)$$

and

$$\delta_{qr} = \begin{cases} 1, & \text{if } q^{-1}(p) < r^{-1}(p), \\ 0, & \text{otherwise.} \end{cases}$$

Analogously we define δ_{rq}, δ_{rp}, δ_{pr}, δ_{pq} and δ_{pq}.

The following theorem derives a criterion for the kth coefficient of Φ_{pqr} to be jumping up or down.

Theorem 3.1. *The value $V(k)$ depends on which one of the sets $\mathcal{A}^p_{j_1} \times \mathcal{A}^q_{j_2} \times \mathcal{A}^r_{j_3}$ contains (a_k, b_k, c_k) in the way described in Table 1. The notation $(j_1 j_2 j_3)$ in the first column means $(a_k, b_k, c_k) \in \mathcal{A}^p_{j_1} \times \mathcal{A}^q_{j_2} \times \mathcal{A}^r_{j_3}$.*

TABLE 1. the values of $\mathrm{oct}(k)$ in dependence on (a_k, b_k, c_k)

$(j_1 j_2 j_3)$	$\mathrm{oct}(k) \sim$	$V(k)$
001	$(0, 1, 1, 1, 1, 2, 2, 2)$	1
002	$(0, \delta_{pq}, \delta_{qp}, 1, 1 + \delta_{qp}, 1 + \delta_{pq}, 1, 2)$	0
003	$(0, 0, 0, 1, 1, 1, 1, 2)$	-1
004	$(0, 0, 0, 1, 1, 1, 0, 1)$	0
011	$(0, 1, 1, 1, 2, 1, 1, 1)$	1
012	$(0, \delta_{pq}, \delta_{qp}, 1, 1 + \delta_{qp}, \delta_{pq}, 1, 1)$	δ_{qp}
013	$(0, 0, 0, 1, 1, 0, 1, 1)$	0
014	$(0, 0, 0, 1, 1, 0, 0, 0)$	0
022	$(0, \delta_{pq} + \delta_{pr} - 1, \delta_{qp}, \delta_{rp}, \delta_{qp} + \delta_{rp}, \delta_{pq}, \delta_{pr}, 1)$	0
023	$(1, \delta_{pr}, 1, 1 + \delta_{pr}, 1 + \delta_{pr}, 1, 1 + \delta_{pr}, 2)$	$-\delta_{rp}$
024	$(1, \delta_{pr}, 1, 1 + \delta_{pr}, 1 + \delta_{pr}, 1, \delta_{pr}, 1)$	0
033	$(1, 0, 1, 1, 1, 1, 1, 2)$	-1
034	$(1, 0, 1, 1, 1, 1, 0, 1)$	0
044	$(1, 0, 1, 1, 1, 0, 0, 0)$	0
111	$(0, 1, 1, 1, 1, 1, 1, 0)$	0
112	$(0, \delta_{pq}, \delta_{qp}, 1, \delta_{qp}, \delta_{pq}, 1, 0)$	0
113	$(0, 0, 0, 1, 0, 0, 1, 0)$	0
114	$(1, 1, 1, 2, 1, 1, 1, 0)$	-1
122	$(1, \delta_{pq} + \delta_{pr}, 1 + \delta_{qp}, 1 + \delta_{rp},$ $\delta_{qp} + \delta_{rp}, 1 + \delta_{pq}, 1 + \delta_{pr}, 1)$	$\delta_{pq}\delta_{pr} - \delta_{qp}\delta_{rp}$
123	$(1, \delta_{pr}, 1, 1 + \delta_{rp}, \delta_{rp}, 1, 1 + \delta_{pr}, 1)$	$\delta_{pr} - \delta_{rp}$
124	$(1, \delta_{pr}, 1, 1 + \delta_{rp}, \delta_{rp}, 1, \delta_{pr}, 0)$	$-\delta_{rp}$
133	$(1, 0, 1, 1, 0, 1, 1, 1)$	0
134	$(1, 0, 1, 1, 0, 1, 0, 0)$	0
144	$(2, 1, 2, 2, 1, 1, 1, 0)$	-1
222	$(1, \delta_{pq} + \delta_{pr}, \delta_{qr} + \delta_{qp}, \delta_{rp} + \delta_{rq},$ $\delta_{qp} + \delta_{rp}, \delta_{rq} + \delta_{pq}, \delta_{pr} + \delta_{qr}, 1)$	0
223	$(1, \delta_{pr}, \delta_{qr}, \delta_{rp} + \delta_{rq}, \delta_{rp}, \delta_{rq}, \delta_{pr} + \delta_{qr}, 1)$	$\delta_{pr}\delta_{qr} - \delta_{rp}\delta_{rq}$
224	$(1, \delta_{pr}, \delta_{qr}, \delta_{rp} + \delta_{rq}, \delta_{rp}, \delta_{rq}, \delta_{pr} + \delta_{qr} - 1, 0)$	0
233	$(1, 0, \delta_{qr}, \delta_{rq}, 0, \delta_{rq}, \delta_{qr}, 1)$	0
234	$(2, 1, 1 + \delta_{qr}, 1 + \delta_{rq}, 1, 1 + \delta_{rq}, \delta_{qr}, 1)$	δ_{rq}
244	$(2, 1, 1 + \delta_{qr}, 1 + \delta_{rq}, 1, \delta_{rq}, \delta_{qr}, 0)$	0
333	$(1, 0, 0, 0, 0, 0, 0, 1)$	0
334	$(2, 1, 1, 1, 1, 1, 0, 1)$	1
344	$(2, 1, 1, 1, 1, 0, 0, 0)$	1

In order to prove Theorem 3.1 we need the following, simple fact.

Lemma 3.2. *If* $(a_k, b_k, c_k) \in \mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_3^r$ *and* $(a_{k'}, b_{k'}, c_{k'}) \in \mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_4^r$, *then*

$$oct(k') \sim (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}+1, F_{k-p-q-r}+1).$$

Similarly, if $k \in \mathcal{A}_1^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$ *and* $k' \in \mathcal{A}_0^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$, *then*

$$oct(k') \sim (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r}-1, F_{k-r-p}, F_{k-p-q}, F_{k-p-q-r}-1).$$

Proof. Let us consider the first situation. By Proposition 2.2 and its symmetric versions, it follows that

$$F_k - F_{k-p} = F_{k'} - F_{k'-p}, \quad F_k - F_{k-q} = F_{k'} - F_{k'-q}, \quad F_k - F_{k-r} = F_{k'} - F_{k'-r}.$$

Then, by Proposition 2.3 and its symmetric versions we have

$$F_k - F_{k-q-r} = F_{k'} - F_{k'-q-r}, \qquad F_k - F_{k-r-p} = F_{k'} - F_{k'-r-p}.$$

Once more we use a symmetric version of Proposition 2.3, which gives the equalities

$$F_k - F_{k-p} - F_{k-q} + F_{k-p-q} = F_{k-r} - F_{k-r-p} - F_{k-q-r} + F_{k-p-q-r} = 1,$$

$$F_{k'} - F_{k'-p} - F_{k'-q} + F_{k'-p-q} = F_{k'-r} - F_{k'-r-p} - F_{k'-q-r} + F_{k'-p-q-r} = 0.$$

Thus the first claim is true. The proof of the second one is similar. $\square$

Now we are ready to prove the main result of this section.

Proof of Theorem 3.1. We determine $oct(k)$ up to adding an integer, so in each row of Table 1 we fixed F_k arbitrarily. First, consider the cases $(j_1 j_2 j_3)$ from Table 1 which are not of the form $(0 \ldots)$ or $(\ldots 4)$. Using Proposition 2.2 and its symmetric versions, we obtain the values of F_{k-p}, F_{k-q}, F_{k-r}. Then, by Proposition 2.3 and its symmetric versions we compute F_{k-q-r}, F_{k-r-p}, F_{k-p-q}. At the end we use Proposition 2.4 to determine $F_{k-p-q-r}$.

We assumed that $\delta_{pq} + \delta_{qp} = 1$ since if $\delta_{pq} = \delta_{qp} = 0$ then $\mathcal{A}_2^r = \emptyset$ and the case is empty. The situation with δ_{rq}, δ_{rq} and δ_{rp}, δ_{pr} is analogous.

Now we can use Lemma 3.2 to compute $oct(k)$ for remaining cases $(j_1 j_2 j_3)$: of form $(0 \ldots)$ and $(\ldots 4)$. After these computations the second column of Table 1 is complete.

It is time to compute $V(k)$, for which we use Proposition 2.5. In rows which does not contain δ's the calculation is straightforward. We consider remaining cases one by one. We write $oct(k) \sim (\ldots)$ if there is the equality up to adding an integer.

(002) It does not matter which one of δ_{pq}, δ_{qp} equals 1, so we assume that $\delta_{pq} = 1$. Then $oct(k) = (0, 1, 0, 1, 1, 2, 1, 2)$ and $V(k) = 0$.

(244) This case is analogous to the previous one.

(112) Again, the value of δ_{pq} is not important and for $\delta_{pq} = 1$ we have $oct(k) \sim (0, 1, 0, 1, 0, 1, 1, 0)$ and $V(k) = 0$.

(233) As before, the value of δ_{qr} has no influence on $V(k)$ and we can assume $\delta_{qr} = 1$. Then $oct(k) \sim (1, 0, 1, 0, 0, 0, 1, 1)$ and $V(k) = 0$.

(012) For $\delta_{pq} = 1$ we have $\mathrm{oct}(k) \sim (0,1,0,1,1,1,1,1)$ and $V(k) = 0$. For $\delta_{qp} = 1$ we have $\mathrm{oct}(k) = (0,0,1,1,2,0,1,1)$ and $V(k) = 1$. Thus $V(k) = \delta_{qp}$.

(234) For $\delta_{qr} = 1$ we have $\mathrm{oct}(k) \sim (2,1,2,1,1,1,1,1)$ and $V(k) = 0$. For $\delta_{rq} = 1$ we have $\mathrm{oct}(k) = (2,1,1,2,1,2,0,1)$ and $V(k) = 1$. So $V(k) = \delta_{rq}$.

(024) If $\delta_{pr} = 0$ then $\mathrm{oct}(k) \sim (1,0,1,1,1,1,0,1)$. If $\delta_{pr} = 1$ then $\mathrm{oct}(k) \sim (1,1,1,2,2,1,1,1)$. In both cases $V(k) = 0$.

(123) If $\delta_{rp} = 1$, then $\mathrm{oct}(k) = (1,0,1,2,1,1,1,1)$ and $V(k) = -1$. If $\delta_{pr} = 1$, then $\mathrm{oct}(k) = (1,1,1,1,0,1,2,1)$ and $V(k) = 1$. So we have $V(k) = \delta_{pr} - \delta_{rp}$.

(023) For $\delta_{rp} = 1$ we have $\mathrm{oct}(k) = (1,0,1,1,1,1,1,2)$ and $V(k) = -1$. For $\delta_{pr} = 1$ we have $\mathrm{oct}(k) \sim (1,1,1,2,2,1,2,2)$ and $V(k) = 0$. Thus $V(k) = -\delta_{rp}$.

(124) If $\delta_{rp} = 1$, then $\mathrm{oct}(k) = (1,0,1,2,1,1,0,0)$ and $V(k) = -1$. When $\delta_{pr} = 1$, we have $\mathrm{oct}(k) \sim (1,1,1,2,2,1,1,1)$ and $V(k) = 0$. So $V(k) = -\delta_{rp}$.

(022) Note that in this case the situation $\delta_{pq} = \delta_{pr} = 0$ is impossible, because then $F_{m-q-r} = F_{m-p} + 3$, contradicting Proposition 2.1. If $\delta_{pq} = \delta_{pr} = 1$, then $\mathrm{oct}(k) \sim (0,1,0,0,0,1,1,1)$ and $V(k) = 0$. If one of δ_{pq}, δ_{pr} equals 1, we assume that $\delta_{pq} = 1$ (it does not matter). Then we have $\mathrm{oct}(k) \sim (0,0,0,1,1,1,0,1)$ and $V(k) = 0$.

(224) This case is analogous to the previous one.

(122) We have the following equalities

$$\mathrm{oct}(k) \sim \begin{cases} (1,2,1,1,0,2,2,1) & \text{if } \delta_{pq} = \delta_{pr} = 1, \\ (1,0,2,2,2,1,1,1) & \text{if } \delta_{qp} = \delta_{rp} = 1, \\ (1,1,1,2,1,2,1,1) & \text{if } \delta_{pq} = \delta_{rp} = 1, \\ (1,1,2,1,1,1,2,1) & \text{if } \delta_{qp} = \delta_{pr} = 1, \end{cases}$$

by which we obtain $V(k) = \delta_{pq}\delta_{pr} - \delta_{qp}\delta_{rp}$.

(223) Similarly as in the previous case, we have

$$\mathrm{oct}(k) \sim \begin{cases} (1,1,1,0,0,0,2,1) & \text{if } \delta_{pr} = \delta_{qr} = 1, \\ (1,0,0,2,1,1,0,1) & \text{if } \delta_{rp} = \delta_{rq} = 1, \\ (1,1,0,1,0,1,1,1) & \text{if } \delta_{pr} = \delta_{rq} = 1, \\ (1,0,1,1,1,0,1,1) & \text{if } \delta_{rp} = \delta_{qr} = 1. \end{cases}$$

We conclude that $V(k) = \delta_{pr}\delta_{qr} - \delta_{rp}\delta_{rq}$.

(222) We have

$$\mathrm{oct}(k) \sim \begin{cases} (1,1,1,1,1,1,1,1) & \text{if } \delta_{pq} = \delta_{qr} = \delta_{rp} = 1, \\ (1,2,1,0,0,1,2,1) & \text{if } \delta_{pq} = \delta_{qr} = \delta_{pr} = 1. \end{cases}$$

In both cases above we have $V(k) = 0$. The remaining ones are symmetric.

Thus we verified all the cases from Table 1 and the proof of Theorem 3.1 is complete. $\qquad\square$

Let us add that there are 125 sets of type $\mathcal{A}^p_{j_1} \times \mathcal{A}^q_{j_2} \times \mathcal{A}^r_{j_3}$. By the symmetry, using Theorem 3.1, we are able to obtain all of them except two. The exceptions are $\mathcal{A}^p_0 \times \mathcal{A}^q_0 \times \mathcal{A}^r_0$ and $\mathcal{A}^p_4 \times \mathcal{A}^q_4 \times \mathcal{A}^r_4$. The next lemma justifies the lack of them in Table 1 by proving that these products are empty.

Lemma 3.3. *Exactly one or two of the following three inequalities*

$$q^{-1}(p) + r^{-1}(p) > p, \quad r^{-1}(q) + p^{-1}(q) > q, \quad p^{-1}(r) + q^{-1}(r) > r$$

hold at the same time.

Proof. Summing the sides of the equality

$$\frac{q^{-1}(p)}{p} + \frac{p^{-1}(q)}{q} = 1 + \frac{1}{pq}$$

with the sides of the symmetric equalities, we receive

$$\frac{q^{-1}(p) + r^{-1}(p)}{p} + \frac{r^{-1}(q) + p^{-1}(q)}{q} + \frac{p^{-1}(r) + q^{-1}(r)}{r} = 3 + \frac{1}{qr} + \frac{1}{rp} + \frac{1}{pq}.$$

Hence the lemma follows. $\square$

4. A FORMULA FOR J_{pqr}

Before we present the announced formula, we need the following notation

$$\alpha_p = \min\{q^{-1}(p), r^{-1}(p), p - q^{-1}(p), p - r^{-1}(p)\}, \quad \beta_p = (\alpha qr)^{-1}(p),$$

and similarly we define α_q, α_r, β_q, β_r. One can easily check that

$$\#\mathcal{A}^p_1 = \#\mathcal{A}^p_3 = \alpha_p, \quad \#\mathcal{A}^p_2 = \beta_p - \alpha_p, \quad \#\mathcal{A}^p_0 + \#\mathcal{A}^p_4 = p - \alpha_p - \beta_p$$

and analogous inequalities hold for sets $\mathcal{A}^q_j$ and $\mathcal{A}^r_j$. Let also

$$\delta_p = \delta_{pq}\delta_{pr} + \delta_{rp}\delta_{qp},$$

similarly δ_q and δ_r. If the first inequality from Lemma 3.3 is the only false or the only true one, then we put

$$R = \alpha_p(q - \alpha_q - \beta_q)(r - \alpha_r - \beta_r).$$

If the only true/false is the second or the third inequality, then we define R analogously. In addition we put

$$S = \sum_{\text{cycl}} \delta_p \alpha_p (\beta_q - \alpha_q)(\beta_r - \alpha_r),$$

$$T = \sum_{\text{perm}} \delta_{qr}(\beta_p - \alpha_p)(\alpha_q \#\mathcal{A}^r_0 + \alpha_r \#\mathcal{A}^q_4),$$

where

$$\sum_{\text{cycl}} f(p, q, r) = f(p, q, r) + f(r, p, q) + f(q, r, p),$$

$$\sum_{\text{perm}} f(p, q, r) = f(p, q, r) + f(r, p, q) + f(q, r, p)$$

$$+ f(r, q, p) + f(p, r, q) + f(q, p, r).$$

Now we are ready to present the main result of this section.

Theorem 4.1. *We have*

$$J_{pqr} = R + S + T + \sum_{cycl} \alpha_p \alpha_q (r - \alpha_r - \beta_r).$$

Proof. In order to make the notation more readable, we put

$$\sigma^{\text{perm}}_{j_1 j_2 j_3} = \sum_{\text{perm}} \#\mathcal{A}^p_{j_1} \#\mathcal{A}^q_{j_2} \#\mathcal{A}^r_{j_3},$$

$$\sigma^{\text{perm}}_{j_1 j_2 j_3}(f(p,q,r)) = \sum_{\text{perm}} f(p,q,r) \#\mathcal{A}^p_{j_1} \#\mathcal{A}^q_{j_2} \#\mathcal{A}^r_{j_3}$$

and analogously

$$\sigma^{\text{cycl}}_{j_1 j_2 j_3} = \sum_{\text{cycl}} \#\mathcal{A}^p_{j_1} \#\mathcal{A}^q_{j_2} \#\mathcal{A}^r_{j_3},$$

$$\sigma^{\text{cycl}}_{j_1 j_2 j_3}(f(p,q,r)) = \sum_{\text{cycl}} f(p,q,r) \#\mathcal{A}^p_{j_1} \#\mathcal{A}^q_{j_2} \#\mathcal{A}^r_{j_3}$$

By Theorem 3.1 we have

$$J_{pqr} = \sigma^{\text{cycl}}_{001} + \sigma^{\text{cycl}}_{011} + \sigma^{\text{cycl}}_{334} + \sigma^{\text{cycl}}_{344} + \sigma^{\text{perm}}_{123}(\delta_{pr})$$
$$+ \sigma^{\text{perm}}_{012}(\delta_{qp}) + \sigma^{\text{perm}}_{234}(\delta_{rq}) + \sigma^{\text{cycl}}_{122}(\delta_{pq}\delta_{pr}) + \sigma^{\text{cycl}}_{223}(\delta_{pr}\delta_{qr}).$$

It is easy to observe that

$$\sigma^{\text{cycl}}_{001} + \sigma^{\text{cycl}}_{344} = \sigma^{\text{cycl}}_{100} + \sigma^{\text{cycl}}_{344} = \sum_{cycl} \alpha_p \big(\#\mathcal{A}^q_0 \#\mathcal{A}^r_0 + \#\mathcal{A}^q_4 \#\mathcal{A}^r_4\big) = R,$$

$$\sigma^{\text{cycl}}_{011} + \sigma^{\text{cycl}}_{334} = \sigma^{\text{cycl}}_{011} + \sigma^{\text{cycl}}_{411} = \sum_{cycl}(p - \alpha_p - \beta_p)\alpha_q \alpha_r.$$

Now we consider sums containing δ's. The equalities above remain true, since if $\delta_{pr} + \delta_{rp} \neq 1$, then the set $\mathcal{A}^q_2$ is empty. We have

$$\sigma^{\text{perm}}_{123}(\delta_{pr}) = \sigma^{\text{cycl}}_{123}(\delta_{pr}) + \sigma^{\text{cycl}}_{321}(\delta_{rp}) = \sum_{cycl} \alpha_p \alpha_q (\beta_r - \alpha_r),$$

$$\sigma^{\text{perm}}_{012}(\delta_{qp}) + \sigma^{\text{perm}}_{234}(\delta_{rq}) = \sigma^{\text{perm}}_{210}(\delta_{qr}) + \sigma^{\text{perm}}_{243}(\delta_{qr}) = T.$$

At last,

$$\sigma^{\text{cycl}}_{122}(\delta_{pq}\delta_{pr}) + \sigma^{\text{cycl}}_{223}(\delta_{pr}\delta_{qr}) = \sigma^{\text{cycl}}_{122}(\delta_{pq}\delta_{pr}) + \sigma^{\text{cycl}}_{322}(\delta_{rp}\delta_{qp}) = S.$$

By summing the received values, we get the thesis.   □

As a consequence of Theorem 4.1 we obtain Theorem 1.1.

Proof of Theorem 1.1. We will use the fact that $ab \geq a+b-1$ for every positive integers a and b. By Theorem 3.1 and the obvious inequality $R, S, T \geq 0$, we have

$$
\begin{aligned}
J_{pqr} &\geq \sum_{\text{cycl}} \alpha_p \alpha_q (r - 2\alpha_r) = \frac{1}{2} \sum_{\text{cycl}} \alpha_p \big(\alpha_q (r - 2\alpha_r) + (q - 2\alpha_q)\alpha_r \big) \\
&\geq \frac{1}{2} \sum_{\text{cycl}} \big(\alpha_q + (r - 2\alpha_r) - 1 + (q - 2\alpha_q) + \alpha_r - 1 \big) \\
&= \frac{1}{2} \sum_{\text{cycl}} (q - \alpha_q - 1 + r - \alpha_r - 1) \geq \frac{1}{2} \sum_{\text{cycl}} \big((q-1)/2 + (r-1)/2 \big) \\
&= (p-1)/2 + (q-1)/2 + (r-1)/2 > (p+q+r)/3 > \sqrt[3]{pqr},
\end{aligned}
$$

which completes the proof. $\square$

5. Polynomials with small J_{pqr}

Proof. Let $q = tp - 1$, where $3 \leq t < q^\varepsilon$ and let $r = 2q + 1 = 2tp - 1$. Then it is not hard to verify that

$$
\begin{array}{lll}
q^{-1}(p) = p - 1, & r^{-1}(q) = 1, & p^{-1}(r) = 2t, \\
r^{-1}(p) = p - 1, & p^{-1}(q) = t, & q^{-1}(r) = 2tp - 3, \\
\alpha_p = 1, & \alpha_q = 1, & \alpha_r = 2, \\
\beta_p = 1, & \beta_q = t, & \beta_r = 2tp - 2t - 1, \\
\mathcal{A}_4^p = \emptyset, & \mathcal{A}_0^q = \emptyset, & \mathcal{A}_4^r = \emptyset,
\end{array}
$$

and $\delta_{rp} = \delta_{pq} = 1$. The remaining δ's from Theorem 4.1 equal 0. Thereby we have

$$
\sum_{\text{cycl}} \alpha_p \alpha_q (r - 2\alpha_r) = (2tp - 5) + 2(tp - 3) + 2(p - 2) < 6q
$$

and

$$
R = (p - \alpha_p - \beta_p)\alpha_q(r - \alpha_r - \beta_r) = (p - 2)(2t - 2) < 2q.
$$

Since $\delta_p = \delta_q = \delta_r = 0$, we have $S = 0$. It remains to evaluate T:

$$
\begin{aligned}
T &= (\beta_q - \alpha_q)(\alpha_r \# \mathcal{A}_0^p + \alpha_p \# \mathcal{A}_4^r) + (\beta_r - \alpha_r)(\alpha_p \# \mathcal{A}_0^q + \alpha_q \# \mathcal{A}_4^p) \\
&= (t - 1)2(p - 2) < 2q.
\end{aligned}
$$

By Theorem 4.1 we have $J_{pqr} < 10q$, while $pqr > q^{3-\varepsilon}$, so the proof is done. $\square$

In a slightly more general class of the so-called inclusion-exclusion polynomials the exponent $1/3$ in Theorem 1.1 is the best possible. We recall that

$$
\Phi_{pqr}(x) = \frac{(1 - x^{pqr})(1 - x^p)(1 - x^q)(1 - x^r)}{(1 - x^{qr})(1 - x^{rp})(1 - x^{pq})(1 - x)}.
$$

If we replace the assumptions that p, q, r are primes by the assumption that they are pairwise coprime, then the formula above defines the inclusion-exclusion polynomial $Q_{\{p,q,r\}}$ (see [2]).

Let us denote by $J_{\{p,q,r\}}$ the number of jumping up coefficients of the the the polynomial $Q_{\{p,q,r\}}$. As long as $p, q, r > 2$, all results of our paper hold also for the polynomial $Q_{\{p,q,r\}}$.

The numbers m, $6m - 1$, $12m - 1$ are pairwise coprime for every positive integer m. Thus we can repeat the argument from the proof of Theorem 1.3 to deduce that

$$J_{\{m,6m-1,12m-1\}} < 10(6m - 1) < 15n^{1/3},$$

where $n = m(6m - 1)(12m - 1)$ and $m \geq 3$. It derives infinitely many ternary inclusion-exclusion polynomials $Q_{\{p,q,r\}}$ for which $J_{\{p,q,r\}} < 15n^{1/3}$, where $n = pqr$.

Acknowledgments

References

[1] G. Bachman, *On the coefficients of ternary cyclotomic polynomials*, J. Number Theory **100** (2003), 104–116.

[2] G. Bachman, *On ternary inclusion-exclusion polynomials*, Integers **10** (2010), 623–638.

[3] P.T. Bateman, C. Pomerance, R.C. Vaughan, *On the size of the coefficients of cyclotomic polynomials*, Coll. Math. Soc. J. Bolyai **34** (1981), 171–202.

[4] M. Beiter, *Magnitude of the coefficients of the cyclotomic polynomial F_{pqr}*, Amer. Math. Monthly **75** (1968), 370–372.

[5] M. Beiter, *Magnitude of the coefficients of the cyclotomic polynomial F_{pqr}*, II, Duke Math. J. **38** (1971), 591–594.

[6] B. Bzdęga, *Bounds on ternary cyclotomic coefficients*, Acta Arith., **144** (2010), 5–16.

[7] B. Bzdęga, *Sparse binary cyclotomic polynomials*, J. Number Theory **132** (2012), 410–413.

[8] B. Bzdęga, *On the height of cyclotomic polynomials*, Acta Arith. **152** (2012), 349–359.

[9] L. Carlitz, *The number of terms in the cyclotomic polynomial $F_{pq}(x)$*, Amer. Math. Monthly **73** (1966), 979–981.

[10] Y. Gallot, P. Moree, *Neighboring ternary cyclotomic coefficients differ by at most one*, J. Ramanujan Math. Soc. **24** (2009), 235–248.

[11] Y. Gallot, P. Moree, *Ternary cyclotomic polynomials having a large coefficient*, J. Reine Angew. Math. **632** (2009), 105–125.

Adam Mickiewicz University, Poznań, Poland
E-mail address: exul@amu.edu.pl